Groups of Prime Numbers' Differences

Michael U. Chassis

Abstract

This paper reports some results of experimental investigation on computer prime numbers in the range until 1 billion. It has been detected that gaps between consecutive gaps (second differences of primes), named variations, constitute regular and repeating irregular groups. Maximal size of such a group of every type of them depends on 4 special characteristics of each type. All numbers constituting maximal sizes of such groups exist in 2 geometrical figures - hexagram and pentagram.

Besides, as a range of primes increased, the ratio of the number of repeating groups of 2 consecutive variations to the number of primes in the range aspires to 100%.

1 Introduction

As Don Zagier has said, "...prime numbers exhibit stunning regularity" (qtd in [Watkins 2002]). But not all of this regularity lies on the very surface, on the level of primes themselves; some of it is hidden deeper. Indeed, some regularity has been discovered on the level of differences between consecutive primes, gaps, constituting dynamic structure of prime numbers allocation in natural numbers series.

It is convenient to subdivide gaps between consecutive primes into two kinds: pregaps and postgaps. The pregap is the difference between current and previous primes $G_n = P_n - P_{n-1}$, and the postgap is the difference between next and current ones $g_n = P_{n+1} - P_n$. So, $g_n \equiv G_{n+1}$.

All gaps are positive. The minimal gap equal 1 between two first primes, 2 and 3, is unique. All other gaps are even; the value of the maximal gap increases infinitely.

New regularities were discovered thanks to having drawn a histogram of gaps as a function of ordinal numbers of primes (in 1966, by pencil-and-paper work in the range until 6000). In this histogram, we immediately see, in an ocean of apparent chaos, islands of some regularity: symmetrical forms, symmetrical each other forms, and repeating ones (see Fig. 1).

These forms are constituted not by consecutive gaps themselves, but by differences between them - *variations* of gaps. The variation is the difference between the postgap and pregap $V_n = g_n - G_n = P_{n+1} - 2P_n + P_{n-1}$. Introduction of variations in prime numbers analysis is proved to be very effective by having discovered a lot of unexpected regularities.

n	P	G	V	Chart of gaps	
1	2				
2	3	1	1	■	
3	5	2	0	■■	
4	7	2	2	■■	
5	11	4	-2	■■■■	
6	13	2	2	■■	
7	17	4	-2	■■■■	
8	19	2	2	■■	
9	23	4	2	■■■■	
10	29	6	-4	■■■■■■	
11	31	2	4	■■	Symmetrical form
12	37	6	-2	■■■■■■	
13	41	4	-2	■■■■	
14	43	2	2	■■	
15	47	4	2	■■■■	
16	53	6	0	■■■■■■	
17	59	6	-4	■■■■■■	
18	61	2	4	■■	
19	67	6	-2	■■■■■■	
20	71	4	-2	■■■■	
21	73	2	4	■■	Symmetrical
22	79	6	-2	■■■■■■	each other
23	83	4	2	■■■■	forms
24	89	6	2	■■■■■■	
25	97	8	-4	■■■■■■■■	
26	101	4	-2	■■■■	
27	103	2	2	■■	
28	107	4	-2	■■■■	
29	109	2	2	■■	
30	113	4	10	■■■■	
31	127	14	-10	■■■■■■■■■■■■■■	
32	131	4	2	■■■■	
33	137	6	-4	■■■■■■	
34	139	2	8	■■	
35	149	10	-8	■■■■■■■■■■	
36	151	2	4	■■	
37	157	6	0	■■■■■■	
38	163	6	-2	■■■■■■	Repeating
39	167	4	2	■■■■	forms
40	173	6	0	■■■■■■	
41	179	6	-4	■■■■■■	
42	181	2	8	■■	
43	191	10	-8	■■■■■■■■■■	
44	193	2	2	■■	
45	197	4		■■■■	

Figure 1: Histogram of gaps. (This histogram has been created with programs written by the author in Java and then formatted with MS Excel.)

There may be some remote analogy between the dynamic structure of discrete array of prime numbers and calculus if consider gap as the firs derivative and variation as the second one. However, each a prime has two gaps, a pregap and a postgap, but an only variation.

2 Regular and Repeating Irregular Groups

2.1 Types of groups

Regular and repeating irregular groups of consecutive variations give the most meaningful new regularities of dynamic structure of prime numbers constituted by their gaps. The size of such a group is the number of consecutive variations in it except of groups having variations equal zero only what a size is the number of equal consecutive gaps.

All regular groups have symmetry, axial or polar (see Fig. 2). Regular groups of any variations are symmetriads (a, b) having a horizontal axis of symmetry on the histogram of gaps and curls (c, d) having polar symmetry. Special ones are stairs (e) having equal variations (special kind of curls) and zigzags (f) having the same absolute values and alternating signs of consecutive variations (special kind of symmetriads of even size or curls of odd size). Finally, variations equal zero (equal gaps) constitute flats (g) (special kind of all previous kinds of regular groups).

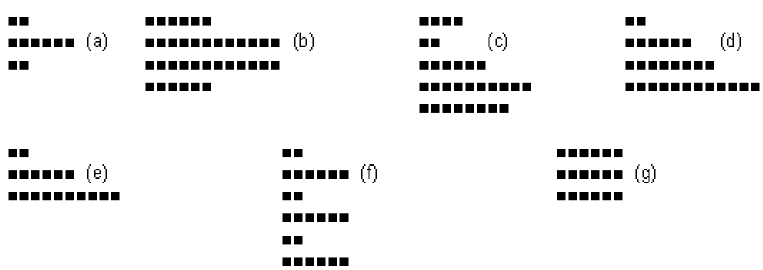

Figure 2: Histograms of regular groups of variations: symmetriads - (a) even, (b) odd; curls - (c) even, (d) odd; (e) stairs; (f) zigzags;(g) flats.

There are for regular groups: for even symmetriads $V_{c-i} = -V_{c+i-1}$; for odd symmetriads $V_{c-i} = -V_{c+i}$, $V_c = 0$, and all V and G are multiple of 6; and for even curls $V_{c-i} = V_{c+i-1}$; for odd curls $V_{c-i} = V_{c+i}$; for stairs $V_i = V_{i+1}$, and zigzags $V_i = -V_{i+1}$; for flats all $V = 0$, all G are multiple of 6 (here V_c and G_c are central variation and pregap).

Repeating groups constitute clusters of 3 kinds: levels, pairs, and foursomes. Flats having the same gaps constitute *levels*. Uniform symmetriads, curls, stairs, and zigzags, having horizontal symmetry each to other on the histogram of gaps, constitute *pairs*. Repeating irregular groups with the same set of variations, having both horizontal and vertical symmetry each to other, do *foursomes*. Some repeating groups form groups of bigger size and some do not.

Irregular repeating groups constituting a foursome are of four kinds (see Fig. 3): groups having the sequence of variations downward (from lesser prime to bigger one) - positive groups (main kind) and negative groups having the same variations with the reversed sign; antigroups having the reversed order of the sequence of variations (upward - from bigger prime to lesser one) - positive antigroups having the same variations with the reversed sign and negative antigroups having the same variations with the same sign.

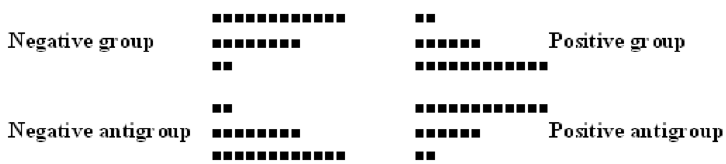

Figure 3: A foursome of irregular repeating groups.

2.2 Maximal sizes of groups

Maximal sizes of regular and repeating irregular groups depend on special characteristics corresponding to their kind. These characteristics are: dynamic coefficient δ, symmetry coefficient σ, size coefficient s, and gaps coefficient γ.

Dynamic coefficient δ is equal 0 if the variations in a group do not change their values, 1 if they change their values and -1 if the signs of the consecutive variations alternate only. Symmetry coefficient σ is equal 0 if the group is irregular, 1 if the variations symmetrical of the center of a regular group have the same sign and -1 if they have the opposite sign; for special types it must be taken $|\sigma| = 1$. Size coefficient s is equal 1 if the size of a group is odd and 2 if it is even; if the type of the group can have $s = 1$ and 2 without changing its peculiarity it must be taken $s = 2 - \sigma$ that is 1 for regular groups and 2 for irregular ones. Gaps coefficient γ is equal 1 if the pregap of a prime is equal to its postgap (one zero variation creates a group of 2 equal consecutive gaps) and 0 if it is not.

In the range until 1,000,000,000, the formula

$$S_{max} = \gamma + \frac{s}{1+\gamma}\{6+6|\sigma|\frac{\delta+\sigma}{1+\gamma+s}+\frac{1}{2}\delta|\sigma|(\delta+|\sigma|)[6^{2-s+\frac{1}{4}(s-1)(s-2\delta)}-6+(2-s)]\}$$

gives values of maximal sizes of every type of regular groups and of repeating irregular groups coinciding with real ones (see Tab.1) found by special programs (in Visual Basic 6.0 for regular groups, in C++ for irregular repeating ones).

It is surprising greatly, all the numbers 2, 3, 6, 7, 10, and 12 there are in the *hexagram* constituted by 2 interlaced triangles and having exactly 6 vertices, 12 points, and 7 subfigures - 6 small triangles and 1 hexagon in the center, and

Type	δ	σ	γ	s		S_{max}	
				Odd	Even	Odd	Even
Symmetriads	1	-1	0	1	2	7	12
Curls	1	1	0	1	2	13(7+6)	10
Flats	0	1	1	1		5(*number of gaps*)	
Zigzags	-1	1	0	1		$6(2 \times 3)$	
Stairs	0	1	0	1		$9(3^2)$	
Irregular repeating groups	1	0	0	2		12(*in the range until* $3,000,000$)	

Table 1: Maximal sizes of groups.

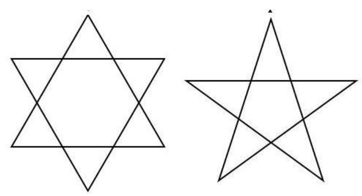

Figure 4: Hexagram and pentagram.

the *pentagram* having 5 vertices, 10 points, and 6 subfigures - 5 small triangles and 1 pentagon in the center.

It was a great opportunity to find the maximal sizes of regular groups gradually in separate consecutive ranges (of 25 millions each on the authors computer). In every of them, the program for sifting primes uses smaller ones found before. Then, with a small overlap with previous range of primes, the programs written by the author calculate variations and find maximal sizes of every type of regular groups in this range. Such way it has succeeded to find them in the range until 1 billion. The fact that they do not change their values from 243,333,233 until 1,000,000,000 gives possibility to suppose they will not increase any more.

Unfortunately, the maximal size of irregular repeating group can be found only from the very beginning because any group may turn out to be repeating. Besides, the program for selecting them is much more complex and, therefore, runs slowly. Because of that, it has succeeded to find the maximal size of irregular repeating group in the range until 3 million only. But, the fact that it found in the range until 36,600 has not increased in the range until 3,000,000 gives possibility to suppose it will not increase in the range of primes until 1,000,000,000 and any more, too.

2.3 Unique groups

Some the biggest groups in the range until 1,000,000,000 may be unigue. Such groups are:

1. The only biggest stars of size 9 with all variations equal -2 (see Fig. 5).
2. The only biggest flat of size 5 with all gaps equal 30 (see Fig. 6).

Figure 5: The unique biggest Stairs

Figure 6: The unique biggest Flat

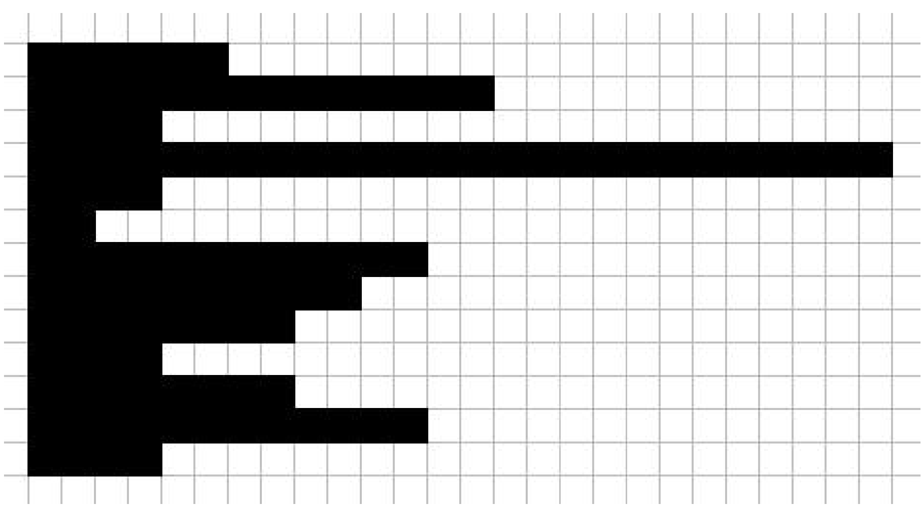

Figure 7: The unique biggest repeating irregular group

3. The only foursome of size 12 found in the range of primes until 3.000.000 may be unique. It consist of two biggest positive repeating irregular groups with the variations +8-10+22-22-2+10-2-2+4+4-8 (see Fig. 7).; all of them are not multiple of 6. But, the beginning pregap of them both is equal 6. The first group begins from the prime $P_{1473} = 12329$, and the second one does from the prime $P_{3868} = 36479$.

2.4 Statistics of regular groups in the range until 1,000,000

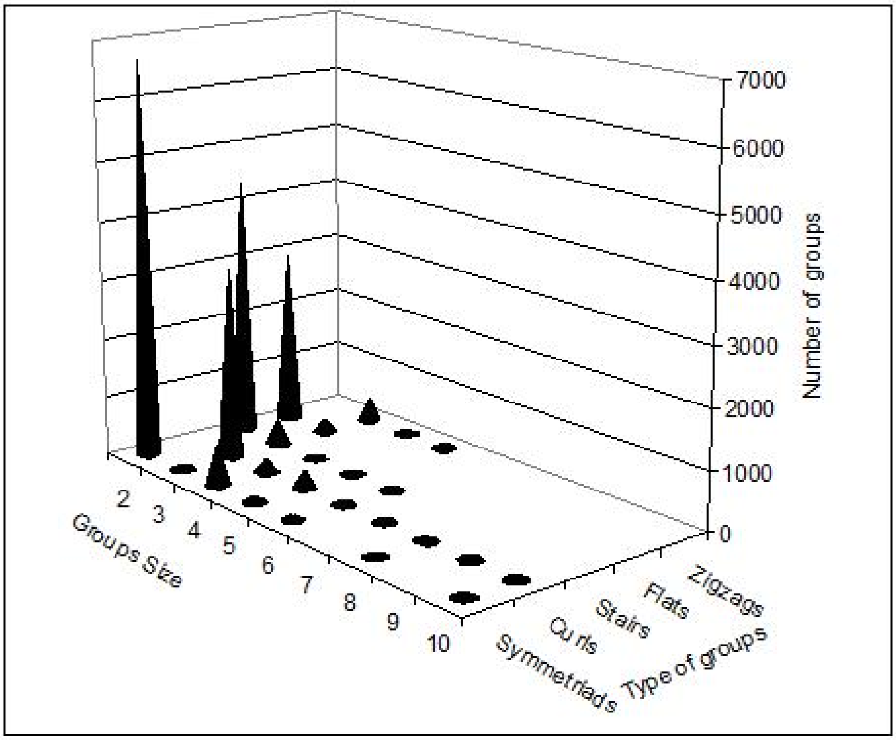

Figure 8: Regular groups in the range until 1,000,000.

Type Size	2	3	4	5	6	7	8	9	10
Symmetriads	6,728	54	109	1	96		14		5
Curls		3,282	246	316	27	24	5	2	1
Stairs	4,427	466	46	5	1				
Flats	2,993	229							
Zigzags		425	51	5					

Table 2: Regular groups in the range until 1,000,000.

Even symmetriads of the size 2 dominate; next are stairs of the size 2, then odd curls of the size 3, and then flats of the size 2.

3 Statistics of numbers of clusters and groups inside them

3.1 Comparative statistics for the ranges until 100,000 and 1,000,000

As the range increases, the ratio of the number of repeating groups of the size of 2 to the number of primes in a range aspires to 100%, and the ratios of the numbers of repeating groups to the number of primes in a range increase for group sizes less or equal 6 and decrease for ones bigger than 6 (see Tab.3).

Size	Number of repeating groups clusters In the range until		Total number of groups in all clusters of the size In the range until		Ratio of repeating groups # to primes #, % In the range until	
	100,000	1,000,000	100,000	1,000,000	100,000	1,000,000
2	257	674	**9,492**	**78,298**	**99**	**99.75**
3	1,119	4,942	8,538	75,170	89	95.76
4	**1,617**	**12,283**	5,547	58,724	58	74.81
5	878	10,220	2,116	27,661	22	35.24
6	278	3,269	591	7,073	6	**9.01**
7	88	653	182	1,331	**2**	1.70
8	31	141	62	283	0.6	0.36
9	13	34	26	68	0.3	0.09
10	5	7	10	14	0.1	0.02
11	**2**	**2**	**4**	**4**	0.04	0.005
12	**1**	**1**	**2**	**2**	0.02	0.0025

Table 3: Repeating groups in the range until 100,000 and 1,000,000.(The range until 100,000 has 9,590 primes, and one until 1,000,000 has 78,498 primes.)

Additionaly, the most diversity of repeating groups is for the size of 4.

3.2 Some more statistics for repeating irregular groups

This is the statistics in the range until 100,000 for any foursomes (see Tab.4)and special types of them.

For some repeating irregular groups are part of ones of the bigger size, other ones constitute independent foursomes (see Tab.5).

Foursomes may have repeating irregular groups from 1 kind only till all 4 kinds. The latter ones are full foursomes (see Tab.6).

And, at least, some independent foursomes are full (see Tab.7).

The biggest foursomes number (maximal diversity of groups) are for any, independent, and full independent ones of the size 4. For full foursomes the biggest number of them is for the size 3.

Size	Number of foursomes	Total number of irregular groups in all foursomes of the group size
2	230	7,812
3	1,059	7,957
4	1,580	5,384
5	871	2,092
6	277	588
7	88	182
8	31	62
9	13	26
10	5	10
11	2	4
12	1	2

Table 4: Number of any foursomes and groups in them

Size	Number of independent foursomes	Total number of groups in all independent foursomes of the group size
2	111	293
3	547	1,297
4	917	2,085
5	493	1,053
6	146	299
7	41	83
8	13	26
9	5	10
10	2	4
11	0	0
12	1	2

Table 5: Number of independent foursomes and groups in them

Size	Number of full foursomes	Total number of groups in all full foursomes of the group size
2	85	6,779
3	180	3,988
4	84	916
5	12	65
6	1	4
7	1	4

Table 6: Number of full foursomes and groups in them

Size	Number of full independent foursomes	Total number of groups in all full independent foursomes of the group size
3	3	15
4	4	22
5	2	11

Table 7: Number of full independent foursomes and groups in them

4 Conclusion

The author reports his results of experimental investigation on computer prime numbers in the range until 1 billion. He studied the regularity of prime numbers on the level of their second differences, variations.

He has detected many regular and repeating irregular groups of consecutive variations, and he has found that a maximal size of each a type of these groups depends on 4 special characteristics of it. The ratio of a number of all repeating groups of 2 consecutive variations to the number of primes in a range of them aspires 100% when the range of primes becomes infinite; so, variations going single are exceptional.

References

[Chassis 2004] Chassis, Michael. Dynamic Structure of Prime Numbers Distribution. 2004. <http://muchassis.org/primes_structure/index1.htm>

[Watkins 2002] Watkins, Matthew R. Prime quotes. 2002. <http://www.maths.ex.ac.uk/~mwatkins/isoc/quotes.htm>.